Mean Square Optimal Hedges Using Higher Order Moments

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Abstract: In this work, we pose and solve a mean square optimal hedging problem that takes higher order moments (or cumulants) into account. We first provide a discrete stochastic dynamics model using a general multinomial lattice, where the first m cumulants are matched over each time step. We then analyze the effect of higher order moments in the underlying asset process on the price of derivative securities. The relationship between the term structure of the volatility smile and smirk and higher order cumulants is illustrated through numerical experiments.

Keywords: mean square optimal hedging, moments, cumulants, derivatives

1 Introduction

It is widely recognized today that there is a non-negligible discrepancy between the Black-Scholes model and real market behavior, which appears as the “smile effect” or “implied volatility smile” in option markets. With empirical evidence that implied volatility increases for in-the-money or out-of-the-money options, option valuation techniques have been extended to more realistic assumptions in a number of ways for the underlying stock process (e.g., [2, 15, 16, 18]) and markets (e.g., [12, 22, 23]). Both theoretical and empirical research on models that exhibit a smile effect and/or heavy tail phenomena became especially active [5, 6, 8, 17, 19, 20, 21], since late 80’s.

In this paper, we model general stochastic dynamics which take any given moments (or cumulants of the underlying stock) into account and apply mean square optimal hedging to evaluate the price of options. A single parameterization of multinomial lattices is demonstrated, where the first m cumulants are matched at each step. Then we analyze the effect of higher order moments in the underlying asset process on the price of derivative securities. The relationship between the term structure of the volatility smile and smirk and higher order cumulants is illustrated through numerical experiments.

2 Mean Square Optimal Hedging

In this paper, we consider a market consisting of two basic securities, a risky asset (or stock) and a risk free asset (or bond), in the time interval \( t \in [0, t_N] \), where traders are allowed to purchase and sell the two basic securities at discrete times \( t_n = n\tau, n = 0, 1, \ldots, N \). Let \( S_n \) be the price of the stock at \( t = t_n \) which satisfies

\[
S_n = S_{n-1}e^{X_n}, \quad n = 1, \ldots, N, \tag{1}
\]

where \( X_n \) is a sequence of i.i.d. random variables whose moments are finite. We will consider a self-financing portfolio which consists of the assets in the underlying market. Let \( r > 0 \) be a fixed interest rate for the risk free asset and \( \Omega_n, n = 0 \ldots N \) be the value of a self-financing portfolio at \( t = t_n \). Then, \( \Omega_n \) satisfies the following difference equation:

\[
\Omega_{n+1} = \Delta_n S_{n+1} + R (\Omega_n - \Delta_n S_n)
\]

\[
= R\Omega_n + \Delta_n (S_{n+1}RS_n), \quad n = 0 \ldots N, \tag{2}
\]

where \( \Delta_n \) is the number of shares of the stock held between \( t \in [t_n, t_{n+1}] \), and \( R := 1 + r \). Finally, let \( C_n, n = 0, 1, \ldots, N \) be the value of a European call option with a strike price \( K \), which pays

\[
C_N = (S_N - K)^+
\]
at maturity \( t_N \).

The objective of mean square optimal hedging is to optimally replicate or hedge the payoff of the call option through a self-financing trading strategy with an adequate initial portfolio value \( \Omega_0 \) at \( t = 0 \). This involves solving the following optimization problem:

Mean Square Optimal Hedging (MSOH)

\[
\min_{\Omega_0, \Omega_0, \ldots, \Omega_{N-1}} E \left[ (C_N - \Omega_N)^2 \right] \quad \tag{3}
\]

subject to the dynamics of the underlying stock and the portfolio, i.e., (1) and (2)

This problem has been studied extensively [8, 9, 10, 12, 22, 23] and can be solved using dynamic programming. Although we only formulate the MSOH problem for a European call option, note that the same approach can be extended to other types of options, including exotics (such as barriers and compounds), and options with time optionality (such as Americans and Bermudans). The dynamic programming algorithm for MSOH only requires a change in the “boundary condition” corresponding to the appropriate option type, and proper discounting to account for the time value of different wealth balance cash flows.

In this paper, we will associate the optimal initial portfolio value of MSOH with the “price” of the option, and assign

\[
C_0 = \Omega_0.
\]
Under this price, \( E(C_N - \Omega_N) = 0 \) is satisfied \[9\], and the objective function in MSOH gives the minimum variance of the hedging error. Therefore, in this situation, the MSOH problem can be thought of as minimizing the risk in the hedge as measured by the variance subject to a zero mean constraint. Although it has been shown that the above “price” can lead to arbitrage opportunities (see the example by Schweizer \[23\]), we will continue to refer to it as a price with the possibility of abuse, in keeping with mean-variance theory.

3 Modeling of stochastic dynamics using moments

In this paper, we will provide a discrete stochastic dynamics model for a stock which takes moment information into account. Recall that the underlying stock dynamics are given by (1) in terms of a sequence of i.i.d. random variables \( X_n \). Equation (1) can be rewritten in log-coordinate as

\[
\ln S_{n+1} - \ln S_n.
\]

Notice that

\[
\ln S_N = \ln S_0 + \sum_{n=0}^{N-1} X_n.
\]

Let us define the \( k \)-the moment of \( X_n \) as

\[
M^{(k)}_n := E(X_n^k).
\]

Also, the \( k \)-th central moment will be denoted by

\[
\hat{M}^{(k)}_n := E\left[(X_n - E(X_n))^k\right].
\]

Here we introduce “cumulants,” where the \( k \)-th cumulant is a polynomial in the moments \( M^{(v)}_n \) with \( v \leq k \) and will be denoted by “\( C^{(v)}_n \).” The first and second cumulants are the mean and variance of \( X_n \), respectively. The third and fourth cumulants are related to skewness and (excess) kurtosis, and are given by

\[
C^{(3)}_n = \hat{M}^{(3)}_n, \quad C^{(4)}_n = \hat{M}^{(4)}_n - 3\hat{M}^{(2)}_n.
\]

The cumulants have an additive property when independent random variables are summed. For example, the \( m \)-th cumulant of \( \sum_{n=0}^{N-1} X_n \) is just the sum of the \( m \)-th cumulants of \( X_n \) for \( n = 0, \ldots, N-1 \). Moreover, a Gaussian distribution is characterized by the fact that all cumulants of order larger than two are identically zero. Therefore the non zero cumulants (higher than order two) can be interpreted as a measure of the difference between a given distribution and a Gaussian.

In the next section, we will present a general description of a random walk which takes any given moments (or cumulants) into consideration on a multinomial lattice. Before stating the result, it will be useful to compute moments of some fundamental stochastic processes as follows:

3.1 Geometric Brownian motion

If the stock follows a geometric Brownian motion, \( S_n \) evolves according to

\[
S_n = S_0 e^{\nu t_n + \sigma B_n}, \quad n = 0, \ldots, N,
\]

where \( B_n \) is a (discrete time) Brownian motion, and \( \nu \) and \( \sigma \) are constant. Equation (6) can be rewritten as

\[
S_n = S_{n-1} e^{X_n}, \quad n = 1, \ldots, N,
\]

where \( X_n \) is a sequence of i.i.d. random variables and each \( X_n \) satisfies

\[
X_n \sim N(\nu \tau, \sigma^2 \tau).
\]

Therefore, it holds that

\[
C^{(1)}_n = \nu \tau, \quad C^{(2)}_n = \sigma^2 \tau, \quad C^{(k)}_n = 0, \quad k \geq 3,
\]

for \( n = 1, \ldots, N \).

3.2 Jump-diffusion models

It is perhaps more convenient to introduce a jump-diffusion model in discrete time as

\[
S_n = S_{n-1} e^{X_n}, \quad n = 1, \ldots, N,
\]

where \( X_n \) consists of a Gaussian random variable \( G_n \) and a jump portion \( J_n Q_n \), i.e.,

\[
X_n = G_n + J_n Q_n, \quad n = 0, \ldots, N.
\]

Here \( G_n \), \( J_n \) and \( Q_n \) are independent, and

\[
G_n \sim N(\nu \tau, \sigma^2 \tau).
\]

\( Q_n \) is a random variable which becomes either 1 with probability \( \lambda \tau \) or 0 with probability \( 1 - \lambda \tau \) for some \( \lambda > 0 \). \( J_n \) determines the jump size which we assume follows another Gaussian distribution

\[
J_n \sim N(\nu J \tau, \sigma^2 J \tau).
\]

To compute the \( k \)-th order cumulant, one can use additive property of cumulants and a recursion relation between cumulants and moments, and finally obtain

\[
C^{(1)}_n = \nu \tau + \lambda \tau \nu J, \quad C^{(2)}_n = \sigma^2 \tau + \lambda \tau \sigma^2 J
\]

\[
C^{(k)}_n = \lambda \tau \nu E(J_k), \quad k \geq 3.
\]

3.3 Stochastic volatility and multidimensional market cases

For multidimensional cases such as a stochastic volatility model, we are unable to write the process as in (1) exactly using i.i.d. random variables \( X_n \). Instead, we can approximate the underlying process based on moment information. For instance, one can derive the moments (or cumulants) of the log-stock return, i.e., \( \ln S_N - \ln S_0 \), using the Kolmogorov Backward Equation for a stochastic volatility model (see \[4\] and references therein), and construct i.i.d. random variables such that the sum of these random variables has the same moments as \( \ln S_N - \ln S_0 \).
4 Multinomial lattices with moments

We will present a general description of a random walk on a multinomial lattice with moments (or cumulants). Suppose that \( u_n \) and \( d_n \) satisfy \( u_n > d_n > 0 \). Then a multinomial tree with \( L \) branches at each node is given by

\[
S_{n+1} = u_n^{L-l} d_n^{l-1} S_n, \quad l = 1, \ldots, L, \tag{15}
\]

where \( p_l, \ l = 1, \ldots, L \) are the corresponding probabilities which satisfy

\[
p_1 + \cdots + p_L = 1. \tag{16}
\]

To make the multinomial tree recombine, we further assume that \( u_n/d_n = c \) for all \( n = 0, \ldots, N - 1 \) for some constant \( c > 1 \). One can verify that the process in (15) consists of a lattice (or a recombining multinomial tree), where the stock may achieve \( n(L-1)+1 \) possible prices at time \( t = n \), \( n = 0, \ldots, N \). For example, in the case of \( u_n = u \) and \( d_n = d \) for all \( n = 1, \ldots, N-1 \), the price of the stock at the \( k \)-th node from the top of the lattice is given by

\[
S_n^{(k)} = u^{n(L-1)+1-k} d^k S_0, \quad k = 1, \ldots, n(L-1)+1. \tag{17}
\]

4.1 Parameterization for Multinomial Lattices with Cumulants

We will provide a parameterization of multinomial lattice random walks which take cumulants into account. Suppose that \( \nu_n \tau \) is the mean of \( X_n \), i.e., the first cumulant (mean) of \( X_n \)

\[
C^{(1)}_n = \mathbb{E}(X_n) = \nu_n \tau. \tag{18}
\]

and let

\[
u_n := \exp \left( \frac{\nu_n \tau}{L-1} + \alpha \sqrt{\tau} \right),
\]

\[
d_n := \exp \left( \frac{\nu_n \tau}{L-1} - \alpha \sqrt{\tau} \right), \tag{19}
\]

where \( \alpha > 0 \) is some constant. One can readily see that \( u_n/d_n \) is constant for all \( n = 0, \ldots, N-1 \) if \( \alpha \) is fixed. With these choices for \( u_n \) and \( d_n \), \( X_n \) may be computed as

\[
X_n = \ln S_{n+1} - \ln S_n = \nu_n \tau + (L-2l+1) \alpha \sqrt{\tau}.
\]

In this case,

\[
\sum_{l=1}^{L} p_l(L-2l+1) = 0 \tag{20}
\]

must hold, and the \( k \)-th central moment \( \hat{M}^{(k)}_n \) is given by

\[
\hat{M}^{(k)}_n = \mathbb{E} \left[ (X_n - \nu_n \tau)^k \right]
\]

\[
= (\alpha \sqrt{\tau})^k \sum_{l=1}^{L} p_l(L-2l+1)^k, \quad k \geq 2. \tag{21}
\]

Note that the second through fourth cumulants are computed by \( C^{(2)}_n = \hat{M}^{(2)}_n \) and the formulas in (25).

4.1.1 Binomial Lattice Case:

We first consider the case of \( L = 2 \), i.e., the binomial lattice case. Since there are already two constraints for the probabilities \( p_1 \) and \( p_2 \), i.e.,

\[
p_1 + p_2 = 1, \quad \sum_{l=1}^{2} p_l(L-2l+1) = p_1 - p_2 = 0,
\]

we obtain \( p_1 = p_2 = 1/2 \). Suppose that the variance of \( X_n \) is given by \( \sigma_n^2 \tau \). This condition restricts \( \alpha = 1 \) and \( \sigma_n \) to be constant, i.e., \( \sigma_n = \sigma (n = 0, \ldots, N-1) \), and we have the well-known binomial lattice formula provided in [14] (see also the original work of [3]):

\[
u_n = \exp (\nu_n \tau + \sigma \sqrt{\tau}), \quad d_n = \exp (\nu_n \tau - \sigma \sqrt{\tau})
\]

\[
p_1 = p_2 = 1/2. \tag{22}
\]

4.1.2 Trinomial Lattice Case:

In the case of a trinomial lattice, i.e., \( L = 3 \), we have one more parameter \( p_3 \), and this allows us to take local volatility information into account, i.e., the second cumulant. Suppose that the second cumulant (i.e., variance) of \( X_n \) is given by \( \sigma_n^2 \tau \). In this case, we have

\[
p_1 + p_2 + p_3 = 1, \quad 2p_1 - 2p_2 = 0, \quad 4p_1 + 4p_2 = \frac{\sigma_n^2}{\alpha^2} \tag{23}
\]

where the second and third equations are obtained from (20) and (21), respectively. By solving (23) with respect to \( p_1, p_2 \) and \( p_3 \), we find

\[
[p_1, p_2, p_3] = \left[ \frac{\sigma_n^2}{8\alpha^2}, 1 - \frac{\sigma_n^2}{4\alpha^2}, \frac{\sigma_n^2}{8\alpha^2} \right].
\]

To guarantee that these probabilities are positive, \( \alpha \) must satisfy \( \sigma/2 < \alpha \). If \( \sigma_n \) is constant, i.e., \( \sigma_n = \sigma (n = 0, \ldots, N-1) \), one may use \( \alpha = \sqrt{3} \sigma/2 \), which provides a trinomial lattice formula whose up, middle, and down rates and corresponding probabilities are given by

\[
u_n = \exp (\nu_n \tau + \sigma \sqrt{3\tau}), \quad d_n = \exp (\nu_n \tau - \sqrt{3\tau})
\]

\[
u_n d_n = \exp (\nu_n \tau), \quad [p_1, p_2, p_3] = [1/6, 2/3, 1/6].
\]

This also corresponds to a well known finite difference scheme (see e.g., [13]).

If \( \sigma_n \) is a function of \( (S_n, n) \), i.e., \( \sigma_n = \sigma(S_n, n) \), the above formula can be modified by writing \( \sigma_n \) in terms of a nominal value \( \hat{\sigma} \) as

\[
\sigma_n = (1 + \delta_n) \hat{\sigma}.
\]

Let \( \alpha \) be chosen as \( \alpha = \sqrt{3} \hat{\sigma}/2 \). Then the up, middle, and down probabilities are given as

\[
[p_1, p_2, p_3] = \left[ \frac{(1 + \delta_n)^2}{6}, 1 - \frac{(1 + \delta_n)^2}{3}, \frac{(1 + \delta_n)^2}{6} \right].
\]

Note that the probabilities are positive as long as \(-\sqrt{3} - 1 < \delta_n < \sqrt{3} - 1 \).
4.1.3 Multinomial Lattice Case:

Now, we show a general lattice case where we have any given moments. Given the first $m$ moments, there are $m + 1$ constraints for $L$ plus 1 unknown parameters, $p_1, \ldots, p_L$, and $\alpha$. If $\alpha > 0$ is fixed a priori, $p_1, \ldots, p_L$ can be computed by solving $m + 1$ linear equations. Therefore, we need at least $L = m + 1$ branches to guarantee the existence of a feasible solution. In this case, $p_1, \ldots, p_L$ can be parameterized as a function of $\alpha$ given below. Finally, $\alpha > 0$ may be adjusted such that all the probabilities are positive. Here we provide a parameterization for up to the 4-th moment (or cumulant), but the extension to the cases with even higher order moments is straightforward.

Suppose that we have third cumulant information corresponding to skewness, in addition to the first and the second cumulants. This imposes

$$C^{(3)}_n = s_n \tau \left( \sigma_n \sqrt{\tau} \right)^3 = (\alpha \sqrt{\tau})^3 \sum_{i=1}^L p_i \left( L - 2l + 1 \right)^3,$$

where $s_n \tau$ is the skewness of $X_n$. Let $L = 4$, and solve four linear equations for the probabilities $p_1, p_2, p_3, p_4$. Then, we obtain

$$[p_1, p_2, p_3, p_4] = \frac{1}{16} \left[ -1 + \frac{\sigma_n^2}{\alpha^2} \left( 1 + \frac{s_n \tau \sigma_n}{3 \alpha} \right), 9 - \frac{\sigma_n^2}{\alpha^2} \left( 1 + \frac{s_n \tau \sigma_n}{\alpha} \right), 9 + \frac{\sigma_n^2}{\alpha^2} \left( 1 - \frac{s_n \tau \sigma_n}{\alpha} \right), -1 + \frac{\sigma_n^2}{\alpha^2} \left( 1 - \frac{s_n \tau \sigma_n}{3 \alpha} \right) \right].$$

If $\sigma_n$ is constant, i.e., $\sigma_n = \sigma \ (n = 0, \ldots, N - 1)$, the choice $\alpha = \sigma/2$ results in the following formulas:

$$u_3^3 = \exp \left( \nu_n \tau + \frac{3 \sigma}{\sqrt{\tau}} \right), \quad u_n^3 d_n = \exp \left( \nu_n \tau + \frac{\sigma}{2 \sqrt{\tau}} \right),$$

$$u_n d_n^2 = \exp \left( \nu_n \tau - \frac{\sigma}{2 \sqrt{\tau}} \right), \quad d_n^3 = \exp \left( \nu_n \tau - \frac{3 \sigma}{2 \sqrt{\tau}} \right),$$

$$[p_1, p_2, p_3, p_4] = \left[ \frac{3}{16} + \frac{1}{6} s_n \tau, \frac{5}{16} - \frac{s_n \tau}{8}, \frac{5}{16} + s_n \tau, \frac{3}{16} - \frac{1}{6} s_n \tau \right].$$

If we would like to match the 4th cumulant or “kurtosis,” we may introduce a multinomial lattice with five branches, i.e., $L = 5$. Let $\kappa \tau$ denote the kurtosis of $X_n$. Then we have

$$C^{(4)}_4 = \kappa_n \tau \left( \sigma_n \sqrt{\tau} \right)^4$$

$$= \alpha^4 \tau^4 \sum_{i=1}^5 p_i \left( 6 - 2l \right)^4 - 3 \left( \sigma_n \sqrt{\tau} \right)^4,$$

as an additional constraint. In this case, the probabilities $p_1, p_2, p_3, p_4, p_5$ can be calculated through the solution of five linear equations, and are given by

$$[p_1, p_2, p_3, p_4, p_5] = \frac{1}{96} \left[ \begin{array}{c} \sigma_n^2 \left( -1 + \frac{s_n \tau \sigma_n}{\alpha} + \frac{\sigma_n^2}{4 \alpha^2} \left( 3 + \kappa_n \tau \right) \right), \\
\frac{2}{\alpha^2} \left( 16 - \frac{2 s_n \tau \sigma_n}{\alpha} - \frac{\sigma_n^2}{\alpha} \left( 3 + \kappa_n \tau \right) \right), \\
\frac{3}{2} \left( 64 + \frac{\sigma_n^2}{\alpha^2} \left( -20 + \frac{\sigma_n^2}{\alpha^2} \left( 3 + \kappa_n \tau \right) \right) \right), \\
\frac{2}{\alpha^2} \left( 16 - \frac{2 s_n \tau \sigma_n}{\alpha} - \frac{\sigma_n^2}{2 \alpha} \left( 3 + \kappa_n \tau \right) \right), \\
\frac{2}{\alpha^2} \left( -1 + \frac{s_n \tau \sigma_n}{\alpha} + \frac{\sigma_n^2}{4 \alpha^2} \left( 3 + \kappa_n \tau \right) \right) \right].$$

To understand the effect of kurtosis, assume that $s_n = 0$ and $\sigma_n = \sigma \ (n = 0, \ldots, N - 1)$, then we obtain

$$[p_1, p_2, p_3, p_4, p_5] = \frac{1}{96} \left[ \begin{array}{c} \sigma^2 \left( -1 + \frac{\sigma}{\alpha^2} \left( 3 + \kappa \tau \right) \right), \\
\frac{2}{\alpha^2} \left( 16 - \frac{2 \sigma}{\alpha^2} \left( 3 + \kappa \tau \right) \right), \\
\frac{3}{2} \left( 64 + \frac{\sigma^2}{\alpha^2} \left( -20 + \frac{\sigma^2}{\alpha^2} \left( 3 + \kappa \tau \right) \right) \right), \\
\frac{2}{\alpha^2} \left( 16 - \frac{2 \sigma}{\alpha^2} \left( 3 + \kappa \tau \right) \right), \\
\frac{2}{\alpha^2} \left( -1 + \frac{\sigma}{\alpha^2} + \frac{\sigma^2}{4 \alpha^2} \left( 3 + \kappa \tau \right) \right) \right].$$

In this case, all the probabilities are positive if

$$\sigma \sqrt{3 + \kappa \tau} < \alpha < \frac{\sigma \sqrt{3 + \kappa \tau}}{2}.$$

Furthermore, if we choose $\alpha = \sigma / \sqrt{2}$, then the above probabilities reduce to

$$[p_1, p_2, p_3, p_4, p_5] = \left[ \frac{1 + \kappa \tau}{96}, \frac{5 - \kappa \tau}{24}, \frac{9 + \kappa \tau}{16}, \frac{5 - \kappa \tau}{24}, \frac{1 + \kappa \tau}{96} \right]. \quad (24)$$

The up-down rates corresponding to five branches can be calculated as

$$u_n^4 = \exp \left( \nu_n \tau + 2 \sigma \sqrt{2 \tau} \right),$$

$$u_n^3 d_n = \exp \left( \nu_n \tau + \sigma \sqrt{2 \tau} \right),$$

$$u_n^2 d_n^2 = \exp \left( \nu_n \tau \right),$$

$$u_n d_n^3 = \exp \left( \nu_n \tau - \sigma \sqrt{2 \tau} \right),$$

$$d_n^4 = \exp \left( \nu_n \tau - 2 \sigma \sqrt{2 \tau} \right).$$

We first notice that the probabilities are symmetric, i.e., $p_1 = p_5$ and $p_2 = p_4$. In this formulation, $p_1, p_3$ and $p_5$
increase with larger kurtosis. On the other hand, $p_2$ and $p_4$ decrease if kurtosis increases. Therefore, this confirms that the probability distribution of $X_n$ becomes heavy tailed under positive kurtosis.

If skewness is not zero, the formulation in (24) becomes

$$[p_1, p_2, p_3, p_4, p_5] = \left[ \frac{1 + \kappa \tau + 2\sqrt{2}s_n\tau}{96}, \frac{5 - \kappa \tau - \sqrt{2}s_n\tau}{24}, \frac{9 + \kappa \tau - 5\kappa \tau + \sqrt{2}s_n\tau}{16}, \frac{1 + \kappa \tau - 2\sqrt{2}s_n\tau}{96} \right]$$

with the choice of $\alpha = \sigma / \sqrt{2}$. In this case, we readily see that the probabilities are not symmetric if $s_n \neq 0$. Moreover, positive (negative) skewness causes $p_1$ and $p_4$ to increase (decrease), and the corresponding probabilities $p_5$ and $p_2$ to decrease (increase) by an equal amount.

5 Illustrative Examples

Here we provide some numerical examples to demonstrate the effect of higher order moments on the implied volatility smile. We use stock index data downloaded from the Chicago Mercantile Exchange. Fig. 1 shows the cumulative distribution of daily log-returns of the stock index from January 1998 to December 2000. The sample mean, standard deviation, skewness and kurtosis are computed as follows:

<table>
<thead>
<tr>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0007</td>
<td>0.0089</td>
<td>-0.3923</td>
<td>3.8207</td>
</tr>
</tbody>
</table>

![Figure 1: Cumulative distribution of daily log-return](image)

With these statistics, we solved the mean square optimal hedging problem to compute the price of a European call option. We first note that the standard model (i.e., the Black-Scholes model) uses the information up to the second moment (i.e., standard deviation) only. We compare the MSOH solution with higher order cumulants with the standard Black-Scholes solution. To understand the difference between the two, we computed the implied volatility for European call options with different maturities and strike prices. Figs. 2–5 are our numerical results, where the implied volatilities are plotted versus the strike prices normalized by the initial price of stock index (i.e., $K/S(0)$). Each figure has a different time to expiration: Fig. 2 has an expiration of 10 days, Fig. 3 20 days, Fig. 4 40 days, and Fig. 5 80 days. Note that the dashed line in each figure is a constant volatility corresponding to the Black-Scholes solution.

First, we note that the smile effect is most clearly observed when the maturity is shortest. The smile effect slowly disappears as we have longer maturities, but a smirk effect remains. This can be explained using the term structure of skewness vs. smirk and the term structure of kurtosis vs. smile as follows.

Let $c_k$ be the $k$-th order cumulant of daily log-returns of the stock index, and let $s$ and $\kappa$ be the corresponding skewness and kurtosis, respectively. Note that the first and second order cumulants are the mean and variance, respectively. Moreover, skewness and kurtosis are functions of cumulants, which are given as follows:

$$s = \frac{c_3}{c_2^{3/2}}, \quad \kappa = \frac{c_4}{c_2} \tag{25}$$

Now, assume that the log-returns of the stock index at each day are independent. Since the cumulants have the additive property when independent random variables are summed, the $k$-th order cumulant of $\sum_{n=0}^{N-1} X_n$, is just the sum of the $m$-th cumulants of $X_n$ for $n = 0, \ldots, N - 1$, i.e.,

$$N c_k, \quad k = 1, 2, \ldots, \tag{26}$$

If we substitute equation (26) to (5), we obtain

$$\text{“} N \text{ day skewness”} = \frac{N c_3}{(N c_2)^{\frac{3}{2}}} = \frac{s}{N^{1/2}}, \quad \text{(27)}$$

$$\text{“} N \text{ day kurtosis”} = \frac{N c_4}{(N c_2)^2} = \frac{\kappa}{N}. \quad \text{(28)}$$

Equations (27) and (28) provide the term structure of skewness and kurtosis for $N$ days with respect to daily skewness and kurtosis.

In the Black-Scholes setting where the distribution of log-stock return is given by a Gaussian distribution, skewness and kurtosis are both zero and the implied volatility is constant. On the other hand, non-zero skewness and kurtosis may increase the smirk and the smile effects as observed from our numerical experiments. Note that this result is consistent with the one described in [4] using the risk neutral probability measure, whereas our numerical experiments indicate that the similar effect is observed in the MSOH problem setting.

6 Conclusion

In this paper, we posed and solved a mean square optimal hedging problem that takes higher order moments (or cumulants) into account. We first showed a discrete stochastic dynamics model using a general multinomial lattice, where
the first $m$ cumulants are matched at each step. We then analyzed the effect of these higher order moments in the underlying asset process on the price of derivative securities. The relationship between the term structure of the volatility smile and smirk and higher order cumulants was illustrated through numerical experiments.

REFERENCES


