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Model Predictive Control for Optimal Portfolios with Cointegrated Pairs of Stocks

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***Abstract*— In this paper, we demonstrate a model predictive control (MPC) approach to constructing an optimal portfolio consisting of multiple spreads of cointegrated pairs of stocks. It is shown that a conditional mean-variance (MV) optimization problem with any given prediction horizon may be solved efficiently when the spreads of stocks follow a vector autoregressive (VAR) model. Based on the solution to the conditional MV problem, we can apply an MPC strategy that calculates the conditional MV optimal portfolio with a given prediction horizon at each rebalancing period. We also perform out-of-sample simulations using empirical stock price data from Japan, and examine the effects of the length of the prediction horizon, rebalance intervals, and transaction costs.**

***Index Terms*— Spread portfolio, Cointegrated pairs, Model predictive control, Conditional mean-variance optimization, Empirical simulations**

I. Introduction

Assume that there is a pair of stocks, say stock A and stock B, whose prices tend to move together, and their spread, defined by the price of stock A minus the price of stock B, is constant on average. Usually, the spread stays around the average level, but occasionally, we may observe a significantly wider spread in the market. In this case we should buy stock B and short sell stock A, believing that the spread will converge to the average in some future period. When the spread converges to the average, we then sell stock B, buy stock A, and clear the position, which will result in a profit related to the difference between the spreads. This trading strategy is called "pairs trading" (see, e.g., [9], [11]) and is considered very useful from the practical point of view since it always leads to a positive profit if the spread converges, i.e., its performance does not depend on whether the market is going up or down.

It is fair to say that stock market prices are believed to evolve according to a random walk and thus cannot be predicted. On the other hand, in some stock markets, we can observe that there are pairs of stocks whose spreads tend to stay around some deterministic level, leading to the idea of "cointegration of pairs of stocks" (see, e.g., [4], [6] for the notion of cointegration). If a pair of stocks is cointegrated, their spread is mean reverting, and we can take advantage of this property to construct a portfolio of multiple spreads of cointegrated pairs.

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In this paper, we formulate a portfolio optimization problem of the spreads of cointegrated pairs of stocks, and develop a model predictive control (MPC) approach based on a conditional mean-variance (MV) optimization with a given prediction horizon. To this end, we first express the spread processes as a vector autoregressive (VAR) model, and show that the conditional MV optimal portfolio may be computed efficiently given information up to the current time. Based on the solution to the conditional MV problem, we apply an MPC strategy that calculates the conditional MV optimal portfolio for a given prediction horizon at each step.

The rest of the paper is organized as follows: In Section II, we introduce the definition of cointegrated processes and formulate the MPC portfolio optimization problem for cointegrated pairs of stocks. In Section III, we illustrate how to find cointegrated pairs, and perform out-of-sample simulations using empirical stock price data from Japan. The effects of the length of prediction horizon, rebalance intervals, and transaction costs are all analyzed. Section IV offers some concluding remarks.

II. Portfolio optimization with cointegrated pairs of stocks

A. Cointegrated processes

To introduce a formal definition of cointegration we need to specify integrated processes; A time series $\{X_t\}_{t=0,\ldots,N}$ is said to be integrated of order d, denoted by $X_t \sim I(d)$, if X_t is nonstationary but becomes stationary after differencing d times. In the context of stock markets, even though the price process X_t is a random walk (which is nonstationary), its integrated process may be stationary, i.e., $\Delta X_t := X_t - X_{t-1}$ is a stationary process, and therefore, $X_t \sim I(1)$ holds. Assume that there are two price processes satisfying $X_t \sim I(1)$ and $Y_t \sim I(1)$. If there exists a non-zero constant β such that $X_t - \beta Y_t$ is stationary, then X_t and Y_t are said to be cointegrated. Assuming that X_t and Y_t stand for stock prices at time t, $X_t - \beta Y_t$ may be thought of as the spread realized by investing in one unit of X_t and $-\beta$ units of Y_t. When we have multiple spreads of cointegrated pairs, we can construct a portfolio of spreads and formulate the spread portfolio optimization.

Consider m pairs of stock price processes,

$$\left(X_t^{(i)},\ Y_t^{(i)}\right),\ i = 1,\ldots,m,$$

with each entry being integrated of order 1. Assume that the vector spread process of $S_t^{(i)} := X_t^{(i)} - \beta^{(i)} Y_t^{(i)},\ i = 1,\ldots,m,$ defined by

$$S_t := \left[S_t^{(1)},\ldots,S_t^{(m)}\right]^\top, \tag{1}$$

follows a VAR model,

$$VAR(q):\ \boldsymbol{S}_t=\Phi_1\boldsymbol{S}_{t-1}+\cdots+\Phi_q\boldsymbol{S}_{t-q}+\boldsymbol{e}_t,\ \ \Phi_i\in\Re^{m\times m} \tag{2}$$

where $\boldsymbol{e}_t$ is an m-dimensional white noise with a covariance matrix $\Sigma\in\Re^{m\times m}$, and $\Phi_i\in\Re^{m\times m},\ i=1,\ldots,q$ a coefficient matrix. Without loss of generality, one can assume that $\mathbb{E}[\boldsymbol{e}_t]=0$ by replacing $\boldsymbol{S}_{t-i}$ in (2) with $\boldsymbol{S}_{t-i}-\mu_0$ for $i=0,1,\ldots,q$, where $\mu_0\in\Re^m$ is the unconditional expectation vector of $\boldsymbol{S}_t$. Note that the control technique demonstrated in this paper may be applied in the case where the spread process is given by a vector AR moving average (VARMA) model and can easily be extended to the case of vector cointegration [6], although we omit the details.

B. Construction of optimal portfolios

Suppose that there are m cointegrated pairs of stock prices whose spreads are modeled as in (2). Noting that each spread can be traded directly by taking a long position in one stock and a short position for the other with appropriate shares of stocks, we construct a self-financing portfolio consisting of multiple spreads and a money market account to maximize a risk adjusted total return of the terminal wealth at a future time $T>0$. In this paper, we formulate an MPC problem for any given prediction horizon $\tau\in(0,\ T)$ and solve it repeatedly at each step, instead of solving a dynamic optimization problem for the terminal time $T>0$ directly. Such a dynamic optimization algorithm is provided in [9], where the spreads are given by the continuous time Ornstein-Uhlenbeck (OU) processes [10] in log coordinates.

Let $u_t^{(i)},\ i\in\{1,\ldots,m\}$ be a share (unit) invested in the i-th spread $S_t^{(i)}$ at day t, W_t be a wealth process corresponding to the value of a self-financing portfolio, and r be the risk free interest rate on the money market account applied for one period. Then, the wealth at time $t+\tau$ is given as

$$W_{t+\tau}=\boldsymbol{u}_t^\top\boldsymbol{S}_{t+\tau}+(1+r)^\tau\left(W_t-\boldsymbol{u}_t^\top\boldsymbol{S}_t\right),$$

where

$$\boldsymbol{u}_t:=\left[u_t^{(1)},\ldots,u_t^{(m)}\right]^\top\in\Re^m.$$

In this case, the total return in the wealth from times t to $t+\tau$, denoted by $R_{t,\tau}$, may be described as follows:

$$R_{t,\tau}:=\frac{W_{t+\tau}}{W_t}=(1+r)^\tau+\frac{\boldsymbol{u}_t^\top}{W_t}\left[\boldsymbol{S}_{t+\tau}-(1+r)^\tau\boldsymbol{S}_t\right]. \tag{3}$$

In our MPC setting, we formulate the following maximization problem over the control input $\boldsymbol{u}_t\in\Re^m$ for the given prediction horizon τ:

$$\max_{\boldsymbol{u}_t\in\Re^m}\left\{\mathbb{E}_t\left[R_{t,\tau}\right]-\frac{\gamma}{2}\cdot\mathbb{V}_t\left[R_{t,\tau}\right]\right\} \tag{4}$$

Here, $\gamma>0$ is a risk aversion coefficient, $\mathbb{E}_t$ the conditional expectation given the information up to time t, and $\mathbb{V}_t$ the conditional variance defined as

$$\mathbb{V}_t\left[R_{t,\tau}\right]:=\mathbb{E}_t\left[\left(R_{t,\tau}-\mathbb{E}_t\left[R_{t,\tau}\right]\right)^2\right].$$

Note that the problem (4) can be interpreted as a conditional version of the usual MV portfolio optimization in [8], where conditional mean and variance are used instead of the unconditional ones in the objective function.

If the problem (4) is solvable for any t and τ, we can apply the following MPC algorithm:

MPC algorithm

Step 0: Select $\delta>0$ and $\tau>0$. Subdivide the time interval $[0,\ T]$ as

$$0<\delta<2\delta<\cdots<(N-1)\delta<T\le N\delta.$$

Let $n=0$ and $t=0$.

Step 1: Set $t_n=n\delta$.

1) If $t=t_n$, solve the problem (4) to find the optimal control input $\boldsymbol{u}_n^*$. Set $\boldsymbol{u}_t=\boldsymbol{u}_n^*$ and update $n\leftarrow n+1$.
2) If $t\neq t_n$, set $\boldsymbol{u}_t=\boldsymbol{u}_n^*$.

Step 2: Compute the wealth at time $t+1$ as

$$W_{t+1}=\boldsymbol{u}_t^\top\boldsymbol{S}_{t+1}+(1+r)\left(W_t-\boldsymbol{u}_t^\top\boldsymbol{S}_t\right). \tag{5}$$

Step 3: Update $t\leftarrow t+1$ and repeat from Step 1.

In the above algorithm, δ and $t_n=n\delta\ (n=0,1,\ldots,N-1)$ determine the rebalance interval (which may also be referred to as the control horizon) and rebalance period, respectively. On the other hand, τ in the problem (4) provides a prediction horizon to maximize the conditional MV objective function. These parameters may be chosen arbitrarily, but usually, they satisfy $0<\delta\le\tau\le T$ to guarantee optimality with respect to the prediction horizon at each time interval $[t_n,\ t_n+\delta]$. In particular, the trading strategy given by the MPC algorithm may be referred to as a "Myopic strategy" in the case of $\tau=1$, in which only the single (and the shortest) period prediction is used.

C. Solution method

Let

$$\boldsymbol{v}_t:=\frac{\boldsymbol{u}_t}{W_t}\in\Re^m, \tag{6}$$

and rewrite the total return of the wealth in (3) as

$$R_{t,\tau}=(1+r)^\tau+\boldsymbol{v}_t^\top\left[\boldsymbol{S}_{t+\tau}-(1+r)^\tau\boldsymbol{S}_t\right]. \tag{7}$$

Then, the problem (4) may be solved by maximizing the objective function over $\boldsymbol{v}_t\in\Re^m$. If the spread process is given by a $VAR(q)$ model in (2) (or a general VARMA model), the conditional mean and variance may be computed analytically as shown below.

Noting that the conditional expectation of $R_{t,\tau}$ is given as

$$\mathbb{E}_t\left[R_{t,\tau}\right]=(1+r)^\tau+\boldsymbol{v}_t^\top\left[\mathbb{E}_t\left(\boldsymbol{S}_{t+\tau}\right)-(1+r)^\tau\boldsymbol{S}_t\right] \tag{8}$$

we have

$$R_{t,\tau}-\mathbb{E}_t\left[R_{t,\tau}\right]=\boldsymbol{v}_t^\top\left[\boldsymbol{S}_{t+\tau}-\mathbb{E}_t\left(\boldsymbol{S}_{t+\tau}\right)\right] \tag{9}$$

Since $VAR(q)$ model in (2) has a moving average representation of

$$\boldsymbol{S}_t=\boldsymbol{e}_t+\Psi_1\boldsymbol{e}_{t-1}+\cdots\Psi_{\tau-1}\boldsymbol{e}_{t-(\tau-1)}+\Psi_\tau\boldsymbol{e}_{t-\tau}+\cdots$$

or equivalently,

$$S_{t+\tau} = e_{t+\tau} + \Psi_1 e_{t+\tau-1} + \cdots \Psi_{\tau-1} e_{t+1} + \Psi_\tau e_t + \cdots \quad (10)$$

for some coefficient matrices $\Psi_i \in \Re^{m\times m}$, $i = 1,2,\ldots$, it holds that

$$R_{t,\tau} - \mathbb{E}_t[R_{t,\tau}] = v_t^\top [e_{t+\tau} + \Psi_1 e_{t+\tau-1} + \cdots \Psi_{\tau-1} e_{t+1}]. \quad (11)$$

Therefore, the conditional variance of $R_{t,\tau}$ may be obtained as

$$\mathbb{V}_t[R_{t,\tau}] = v^\top \left(\Sigma + \Psi_1\Sigma\Psi_1^\top + \cdots + \Psi_{\tau-1}\Sigma\Psi_{\tau-1}^\top\right) v. \quad (12)$$

Consequently, the conditional MV optimization is reduced to the following maximization problem:

$$\max_{v_t\in\Re^m} \Big\{ v_t^\top \left[\mathbb{E}_t(S_{t+\tau}) - (1+r)^\tau S_t\right] - \frac{\gamma}{2}\cdot v_t^\top \left(\Sigma + \Psi_1\Sigma\Psi_1^\top + \cdots + \Psi_{\tau-1}\Sigma\Psi_{\tau-1}^\top\right) v_t \Big\} \quad (13)$$

By applying the first order condition to the problem (13), the maximizer, $v_t = v_t^*$, and the optimal portfolio, $u_t = u_t^*$, are, respectively, found to be

$$v_t^* = \frac{1}{\gamma}\left(\Sigma + \Psi_1\Sigma\Psi_1^\top + \cdots + \Psi_{\tau-1}\Sigma\Psi_{\tau-1}^\top\right)^{-1} \times \left[\mathbb{E}_t(S_{t+\tau}) - (1+r_\tau)S_t\right], \quad (14)$$

$$u_t^* = W_t v_t^* \quad (15)$$

D. Parameter specification in the optimal portfolio

It can be confirmed that the conditional expectation of $R_{t,\tau}$ in (8) is computed by recursively applying (2) and is expressed using state variables up to time t, i.e., $S_t,\ldots,S_{t-q}$. To illustrate the parameter specification, consider the case $q = 1$ in (2). In this case, the spread process is given as

$$VAR(1): \ S_t = \Phi_1 S_{t-1} + c + e_t, \quad (16)$$

where $\Phi_1 \in \Re^{m\times m}$ and $c \in \Re^m$. By applying (16) recursively, $S_{t+\tau}$ may be computed as

$$\begin{aligned} S_{t+\tau} &= \Phi_1 S_{t+\tau-1} + c + e_{t+\tau} \\ &= \Phi_1^2 S_{t+\tau-2} + (\Phi_1 + I)c + (\Phi_1 e_{t+\tau-1} + e_{t+\tau}) \\ &= \quad \cdots \quad \cdots \quad \cdots \\ &= \Phi_1^\tau S_t + \left(\Phi_1^{\tau-1} + \cdots + \Phi_1 + I\right)c \\ &\quad + \left(\Phi_1^{\tau-1} e_{t+1} + \cdots + \Phi_1 e_{t+\tau-1} + e_{t+\tau}\right) \end{aligned}$$

Then, conditional expectation of $S_{t+\tau}$ is specified as

$$\mathbb{E}_t(S_{t+\tau}) = \Phi_1^\tau S_t + \left(\Phi_1^{\tau-1} + \cdots + \Phi_1 + I\right)c. \quad (17)$$

Moreover, since

$$\begin{aligned} R_{t,\tau} - \mathbb{E}_t[R_{t,\tau}] &= v_t^\top [S_{t+\tau} - \mathbb{E}_t(S_{t+\tau})] \\ &= v_t^\top \left(\Phi_1^{\tau-1} e_{t+1} + \cdots + \Phi_1 e_{t+\tau-1} + e_{t+\tau}\right) \end{aligned}$$

holds, we have

$$\Psi_i = \Phi_1^i, \ i = 1,\ldots,\tau-1. \quad (18)$$

in (11) and (12). We see that all the required parameter values for the optimal portfolio may be specified once a $VAR(1)$ model is estimated for S_t. Similarly, we can show that the optimal portfolio u_t^* is fully specified by using coefficient parameters in a $VAR(q)$ model and observable state variables.

III. Empirical simulation

A. Cointegration test for pairs of stocks and data period

1) Cointegration test: For finding cointegrated pairs of stocks, one can apply the Engle-Granger (EG) cointegration test [4]. The objective of the Engle-Granger cointegration test is to determine if a pair of stock prices, (X_t, Y_t) with $X_t \sim I(1)$ and $Y_t \sim I(1)$, is cointegrated based on the following two step processes: (1) Regress Y_t with respect to X_t, and (2) test if the residual (i.e., the spread) has a unit root or is stationary. If the null hypothesis of a unit root is rejected to accept the one-sided alternative hypothesis, we conclude that the spread is stationary and that X_t and Y_t are cointegrated.

2) Pairs selection: When applying the EG cointegration test for N stock prices, the number of possible combinations of pairs is $N(N-1)/2$ but in fact we need to check the cointegration twice for each pair by switching dependent and independent variables in the linear regression of Step 1, although the number of cointegrated vectors is at most no more than N (See [7]). Therefore, we consider the following screening procedure to reduce the number of possible pairs:

Screening procedure:

For every pair of stocks, compute the correlation coefficient and the Dickey-Fuller (DF) statistics of the spread. If the absolute value of correlation is below 0.8 or the DF statistics is above a certain critical value, then remove the pair from the candidates.

The DF statistics is a test statistics to check if the given process has a unit root or is stationary using a simple $AR(1)$ model [3]. We may conclude that the more negative the DF statistic is, the stronger the indication of possible cointegration.

Then, we perform the following selection procedure:

Selection procedure:

Sort the pairs according to the order (smallest to largest) of the DF statistic. Apply the EG cointegration test [4] beginning from the top of the list to select cointegrated pairs without the overlap of a company.

In the above procedure, we first choose a pair from the list, apply the EG cointegration test, and take that pair if it is cointegrated. We then remove any pairs that include the same stock from the rest of the list to avoid overlap.

3) Data period: In this paper, we use daily stock price data consisting of the Nikkei 225 for the period of January 2007 – October 2009. Here we choose 10 categories among 35 categories in Nikkei 225[1]. We then perform the pairs selection procedure (i.e., the screening and the selection procedures) in each category. By applying the pairs selection procedure for the stock price data during the given data period, the 16 pairs in Table I are obtained from the categories denoted in the brackets.

In our simulation, we set the lag order in $VAR(q)$ as $q = 1$, i.e., S_t follows (16). We estimate the parameter values $\Phi_1 \in$

[1]The categories are as of May 2009.

TABLE I

PAIRS LIST USING THE DATA IN THE PERIOD OF 2007–2009

Sapporo vs. Asahi Breweries (Foods),
Ajinomoto vs. Kikkoman (Foods),
Nippon Oil vs. Nippon Mining Holdings (Oil & coal products),
Advantest vs. Taiyo Yuden (Electric machinery),
TDK vs. Tokyo Electron (Electric machinery),
Canon vs. Kyocera (Electric machinery),
Minebea vs. Denso (Electric machinery),
Fanuc vs. Toshiba (Electric machinery),
Fuji Electric vs. Sony (Electric machinery),
Honda vs. Nissan (Automotive),
Matsuda vs. Fuji Heavy Industries (Automotive),
Konica Minolta Holdings vs. Nikon (Precision instruments),
West Japan Railway vs. East Japan Railway (Railway/Bus),
NTT Data vs. NTT (Communications),
Tokyo Gas vs. Chubu Electric Power (Gas/Electric power),
Yahoo vs. Konami (Services)

$\Re^{16\times 16}$ and $\boldsymbol{c} \in \Re^{16}$ in (16) based on the in-sample-period of 2007–2008 (two years), and simulate the wealth for the out-of-sample period of 2009 (10 months).

B. Effect of prediction horizon in MPC

First we discuss the relation between the wealth performance and the length of the prediction horizon. As explained in Section II, we perform the MPC algorithm for the 16 pairs with risk aversion coefficient $\gamma = 80$, risk free rate for one day $r = 0.01/245$, and initial wealth $W_0 = 1$. In this simulation, we assume that the rebalance is done everyday, i.e., the rebalance interval is given by $\delta = 1$. Fig. 1 shows the wealth level of the spread portfolio in the out-of-sample period, where the bottom line is the prediction horizon of 1 day, i.e., $\tau = 1$, corresponding to the "Myopic portfolio," the middle line is that of $\tau = 5$, and the upper line is that of $\tau = 20$. We clearly see that the performance of the wealth is improved significantly as the prediction horizon τ is increased.

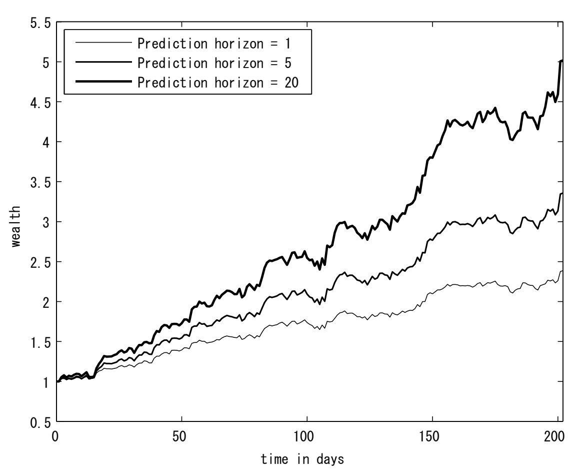

Fig. 1. Wealth in the out-of-sample simulation period for $\tau = 1, 5, 20$

To estimate the efficiency of the spread portfolio, we calculate the Sharpe ratio with respect to different values of the predictive horizon τ, where the Sharpe ratio is the expected excess return normalized per unit of risk defined as $\left(\overline{r_w} - r_f\right)/\sigma(r_w)$, where $\overline{r_w}$ is the mean rate of return of the wealth, $\sigma(r_w)$ the standard deviation of the wealth return, and r_f the risk free interest rate. We estimate the annualized mean rate of return and standard deviation from each sample path of the wealth W_t in the out-of-sample period to obtain $\overline{r_w}$ and $\sigma(r_w)$. Fig. 2 shows the relation between the prediction horizon and the Sharpe ratio, where the annualized Sharpe ratio is calculated for each prediction horizon in the interval between $\tau = 1$ and $\tau = 160$. We see that the annualized Sharpe ratio may be maximized by increasing the prediction horizon until around $\tau = 60$ and it roughly stays at the same level thereafter. Noting that the myopic portfolio is given by the left end point with $\tau = 1$, we conclude that the MPC portfolio with a longer prediction horizon provides better performance in this simulation.

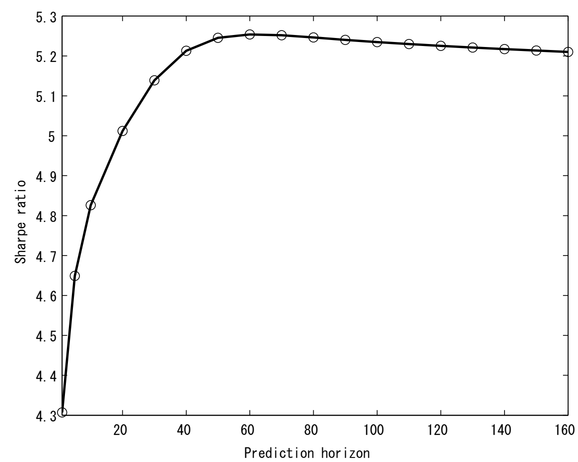

Fig. 2. Prediction horizon vs. Sharpe ratio

C. Effect of rebalance interval

Next, we fix the prediction horizon and vary the rebalance interval δ in the MPC algorithm. Here we set the prediction horizon as $\tau = 60$ which maximizes the Sharpe ratio in the previous simulation. Fig. 3 shows the wealth of the spread portfolio in the out-of-sample period, where the upper line uses a rebalance interval of $\delta = 1$, the middle line $\delta = 5$, and the lower line $\delta = 20$ corresponding to a one month rebalance interval. Although the wealth performance is worse for the larger rebalance interval, it seems that it still performs at a reasonable level even with a rebalance interval of $\delta = 20$.

Similar to the previous simulation, we calculated the annualized Sharpe ratio by changing the rebalance interval from 1 to 40 as shown in Fig. 4. The solid line is the case where the prediction horizon is given by $\tau = 60$, whereas the myopic case of $\tau = 1$ is also plotted as the dashed line. We see that the Sharpe ratio drops almost monotonically with the increase of the rebalance interval for both cases. Although MPC with $\tau = 60$ provides better wealth performance when

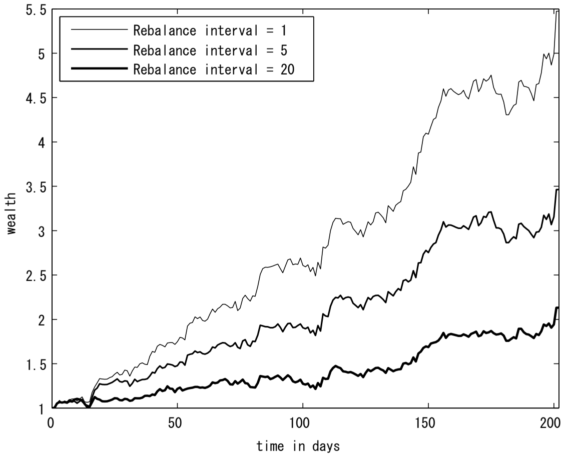

Fig. 3. Wealth in the out-of-sample period for $\delta = 1,5,20$

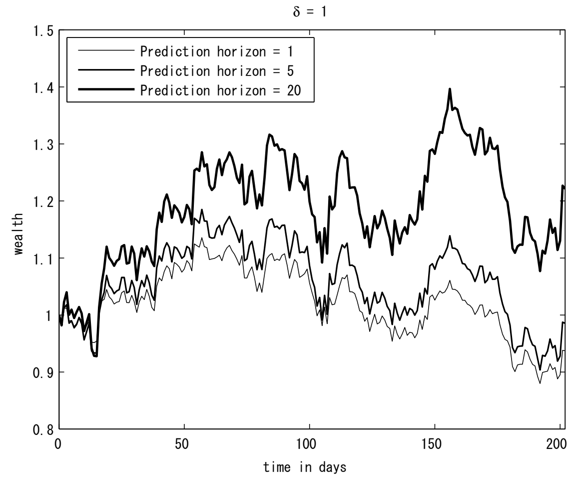

Fig. 5. Wealth in the out-of-sample period for $\tau = 1,5,20$ ($\delta = 1$)

the rebalance interval is between 1 and 30 days, these effects may disappear with rebalance intervals larger than 30 days.

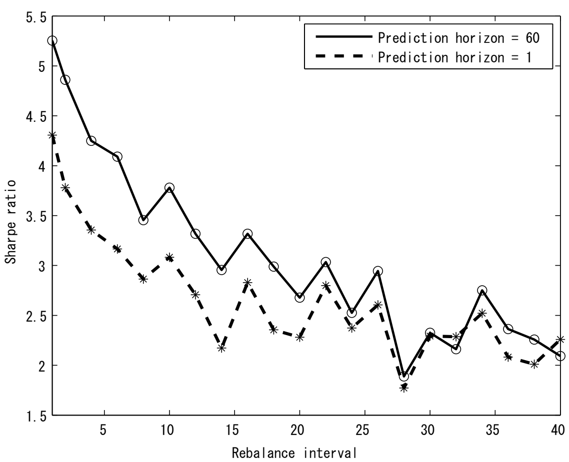

Fig. 4. Rebalance interval vs. Sharpe ratio

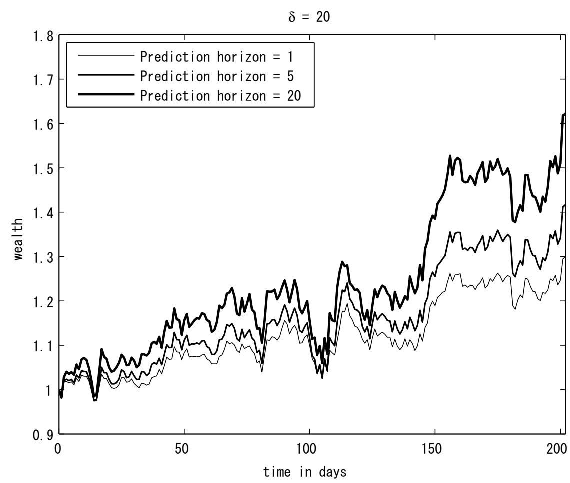

Fig. 6. Wealth in the out-of-sample period for $\tau = 1,5,20$ ($\delta = 20$)

D. *Effect of transaction costs*

Finally, we discuss the effect of transaction costs. Here we assume that there is a proportional transaction cost for each stock trade. That is, when rebalancing the position in a stock we need to pay

$$\rho \times |\text{new account balance} - \text{old account balance}| \qquad (19)$$

at each rebalance period of $t_n = n\delta,\ n = 0,1,\ldots,N-1$ in the MPC algorithm, where ρ is a proportional transaction cost rate. Then, the sum of transaction costs is deducted from the wealth when the transaction is completed. Note that the proportional transaction cost is closely related to a bid-ask spread where one may buy at the ask price and sell at the bid, with the difference between the bid and ask being a transaction cost.

We assumed that the transaction cost rate is given as $\rho = 0.5\%$ (which indicates that the round trip transaction is 1%) and performed the MPC algorithm for the cases of $\tau = 1$ (myopic), $\tau = 5$ and $\tau = 20$. Figs. 5 and 6 show our simulation results, where the rebalance periods are set to $\delta = 1$ and $\delta = 20$, respectively. In these figures, the upper lines are those from the MPC with $\tau = 20$ whereas the bottom lines provide myopic results (i.e., $\tau = 1$). Clearly, incorporating transaction costs makes the wealth performance worse, in particular when $\delta = 1$ under a short prediction horizon such as $\tau = 1$ or $\tau = 5$ in Fig. 5. On the other hand, as shown in Fig. 6, the performance of the wealth for a rebalance period of $\delta = 20$ is not as bad as that for $\delta = 1$, and in this case as well, MPC with $\tau = 20$ performs better than the myopic portfolio.

Figs. 7 and 8 compare Sharpe ratios for different values of the prediction horizon with rebalance intervals of $\delta = 1$ and $\delta = 20$, respectively, where the upper line denotes the Sharpe ratio under $\rho = 0\%$ and the bottom line under $\rho = 0.5\%$.

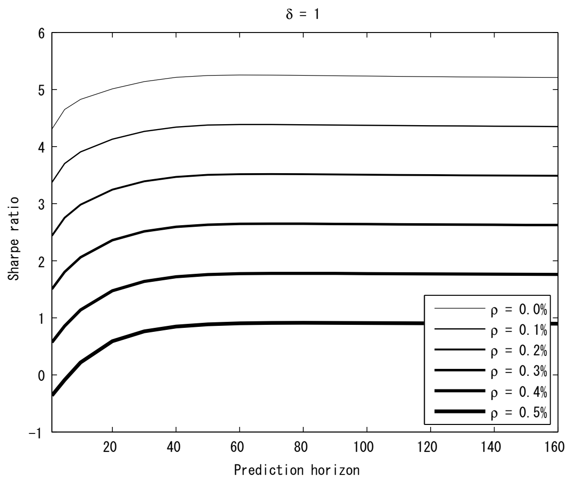

Fig. 7. Prediction horizon vs. Sharpe ratio ($\delta = 1$)

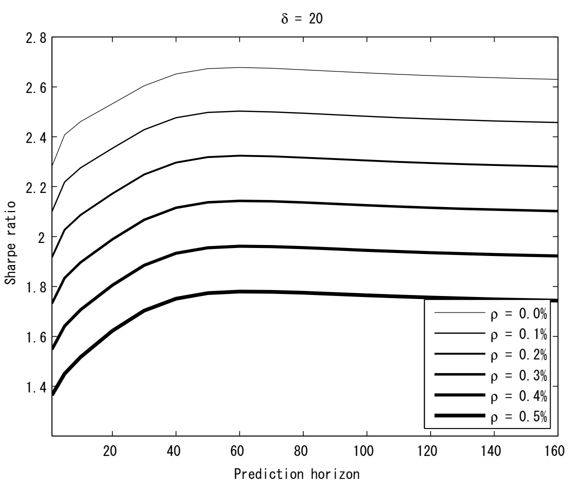

Fig. 8. Prediction horizon vs. Sharpe ratio ($\delta = 20$)

Even with the shortest rebalance period of $\delta = 1$, we see that the wealth performance may be improved by increasing the prediction horizon from $\tau = 1$. In particular, in the case of $\rho = 0.5\%$, the Sharpe ratio was negative for the myopic portfolio but became positive when τ was increased.

The sensitivity of the performance of the wealth with respect to a change in the transaction cost may be observed in Fig. 9, where the Sharpe ratio of the wealth of MPC with $\tau = 60$ is plotted for different values of transaction cost rates when the rebalance interval is $\delta = 1, 5, 10, 20, 40$. We see that the wealth performance decreases with an increase in the transaction cost, similar to previous results, but in the case of $\delta = 1$ the slope is quite steep compared to the others and the Sharpe ratio is the lowest when $\rho = 0.5\%$. This fact may leads to the following observation: If one ignores transaction costs then they might be inclined to use the shortest possible rebalance interval. However, the effect of transaction costs is quite significant if one tries to rebalance frequently. Therefore, it is important to take rebalancing frequency into consideration when transaction costs are present.

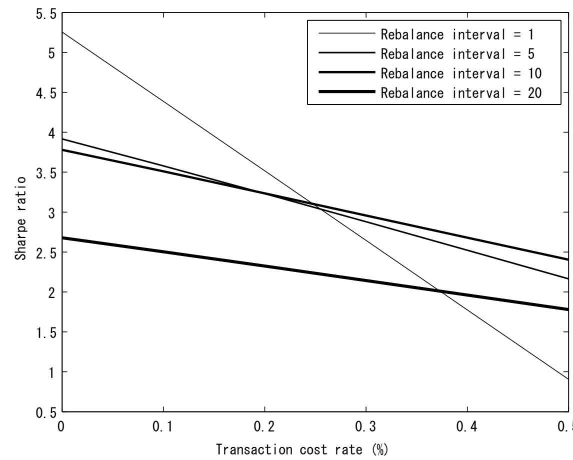

Fig. 9. Transaction cost rate vs. Sharpe ratio ($\tau = 60$)

IV. CONCLUSION

In this paper, we developed an MPC approach to constructing an optimal portfolio consisting of multiple spreads of cointegrated pairs of stocks. At first, we showed that a conditional MV optimization problem under any given prediction horizon may be solved efficiently when the spreads of stocks follow a VAR model. Based on the solution of this conditional MV problem, we utilized an MPC approach that calculated the conditional MV optimal portfolio for a given prediction horizon at each rebalance period. We also performed out-of-sample simulations using empirical stock price data, and examined the effects of the length of prediction horizon, rebalance intervals, and transaction costs.

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